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A Unified Calculus Using the Generalized Bernoulli Polynomials

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We introduce an α-calculus with the help of the generalized Bernoulli polynomials. The parameter α is the order of a Bessel function of the first kind. The differential α-calculus can be put in a general context where the concept of supporting function is an important tool for practical purposes. Our somewhat more restrictive point of view has the advantage of permitting a consistent definition of an α-integral with several interesting properties. It results in the possibility of expressing a remainder, in the aforementioned context, in a completely new form in our case.

1. INTRODUCTION

The α-Bernoulli polynomials $B_{n,\alpha}(x)$ are defined by the generating function

$$\frac{e^{(x-1/2)z}}{g_\alpha\left(\dfrac{iz}{2}\right)} = \sum_{n=0}^{\infty} \frac{B_{n,\alpha}(x)}{n!} z^n, \tag{1}$$

where $g_\alpha(z) := 2^\alpha \Gamma(\alpha+1)(J_\alpha(z)/z^\alpha)$. The series in (1) converges for $|z| < 2|j_1|$, where $j_1 = j_1(\alpha)$ is a zero of $J_\alpha(z)/z^\alpha$ of least modulus. The polynomials $B_{n,\alpha}(x)$, $n = 0, 1, 2, \ldots$, are defined for all complex values of α except for $\alpha = -1, -2, -3, \ldots$. Some general properties of these polynomials are given in [5, 6]. Among other things we proved the following result.

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THEOREM A. *Let α be a complex number, not a negative integer. For each entire function f of exponential type $\tau < 2|j_1|$ which is bounded on the real axis, we have*

$$f(1) = f(0) - \sum_{k=1}^{\infty} \frac{B_{k,\alpha}}{k!} (f^{(k)}(1) - (-1)^k f^{(k)}(0)). \tag{2}$$

Here, $B_{k,\alpha} := B_{k,\alpha}(0)$. We denote by B_τ the class of all entire functions of exponential type τ bounded on the real axis. In the case $\alpha = \frac{1}{2}$ it had been observed that a complex variable can be introduced in (2); the formula

$$f'(z) = \sum_{k=0}^{\infty} \frac{B_k(z)}{k!} (f^{(k)}(1) - f^{(k)}(0)) \tag{3}$$

holds [3, Formula 9.5; 9, p. 220] for all complex numbers z if $f \in B_\tau$, $\tau < 2\pi$. A natural extension of (3) using the α-Bernoulli polynomials leads us to consider a generalized derivative defined by

$$\begin{aligned} b_\alpha(f; z) &:= \sum_{k=0}^{\infty} \frac{((-1)^k - 1)}{k!} B_{k,\alpha} f^{(k)}(z) \\ &= f'(z) + (2\alpha - 1) \left(\frac{1}{48(\alpha+1)} f'''(z) \right. \\ &\quad \left. + \frac{(2\alpha^2 - \alpha - 13)}{7680(\alpha+1)^2 (\alpha+2)} f^{(v)}(z) + \cdots \right) \end{aligned} \tag{4}$$

if the series converges. We see at once from (4) that

$$b_{1/2}(f; z) = f'(z). \tag{5}$$

The limiting case $\alpha \to \infty$ gives, in view of Taylor's formula and the relation [5, p. 309] $\lim_{\alpha \to \infty} B_{k,\alpha}(x) = (x - \frac{1}{2})^k$ with $x = 0$,

$$b_\infty(f; z) = f(z + \tfrac{1}{2}) - f(z - \tfrac{1}{2}). \tag{6}$$

So, the ∞-derivative corresponds exactly to the central finite differences' operator. It will also be seen that

$$\begin{aligned} b_{-1/2}(f; z) &= 4 \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)}{(2k)!} B_{2k} f^{(2k-1)}(z) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{k-1}} \Delta^k f(z) \end{aligned} \tag{7}$$

if both the series converge. Here, $\Delta f(z) := f(z+1) - f(z)$ is the usual finite difference's operator and $\Delta^k f(z) := \Delta^{k-1}(\Delta f(z))$ for $k \geqslant 2$. The first series in (7) converges for each $f \in B_\tau$ with $\tau < \pi$. The second series is defined for more general functions, including $f(z) = \frac{1}{z}$, $z \neq 0, -1, -2, \ldots$.

In general the series in (4) converges for each $f \in B_\tau$ with $\tau < 2\,|j_1|$. This is a consequence of the famous Bernstein's inequality [1], which we use here in the form [2]

$$|f^{(k)}(z)| \leqslant M e^{\tau\,|y|} \tau^k, \tag{8}$$

where $z := x + iy$, $M := \max_{-\infty < x < \infty} |f(x)|$, and the asymptotic relation [5, p. 134]

$$B_{k,\alpha} \sim \frac{-k!(e^{-ij_1} + (-1)^k\, e^{ij_1})}{2^\alpha \Gamma(\alpha+1)(2i)^k\, j_1^{k-\alpha+1} J'_\alpha(j_1)} \tag{9}$$

as $k \to \infty$.

Here we present approximation theoretic applications of the α-derivative (4). This is done by introducing an expansion of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{b_\alpha^{(n)}(f; z_0)}{n!}\, \phi_{n,\alpha}(z - z_0), \tag{10}$$

where $b_\alpha^{(0)}(f; z) := f(z)$ and $b_\alpha^{(n)}(f; z) := b_\alpha(b_\alpha^{(n-1)}(f; z); z)$ for $n = 1, 2, 3, \ldots$. The polynomials $\phi_{n,\alpha}(z)$ are introduced through the recurrence relation

$$\phi_{n,\alpha}(z) = z^n - \sum_{k=0}^{n-1} \frac{b_\alpha^{(k)}(\zeta^n; \zeta = 0)}{k!}\, \phi_{k,\alpha}(z), \tag{11}$$

with $\phi_{0,\alpha}(z) \equiv 1$, $\phi_{1,\alpha}(z) = z$, $\phi_{2,\alpha}(z) = z^2$,

$$\phi_{3,\alpha}(z) = z^3 - \frac{(2\alpha-1)}{8(\alpha+1)}\, z,$$

$$\phi_{4,\alpha}(z) = z^4 - \frac{(2\alpha-1)}{2(\alpha+1)}\, z^2,$$

$$\phi_{5,\alpha}(z) = z^5 - \frac{5(2\alpha-1)}{4(\alpha+1)}\, z^3 + \frac{(2\alpha-1)(18\alpha^2+31\alpha-7)}{64(\alpha+1)^2\,(\alpha+2)}\, z,$$

$$\phi_{6,\alpha}(z) = z^6 - \frac{5(2\alpha-1)}{2(\alpha+1)}\, z^4 + \frac{(2\alpha-1)(4\alpha-1)(16\alpha+31)}{32(\alpha+1)^2\,(\alpha+2)}\, z^2, \qquad \text{etc.}$$

The polynomials $\phi_{k,\alpha}(z)$, $0 \leqslant k \leqslant n$, are linearly independent since they are of exact degree k.

Functions possessing a representation of the form (10) *are said to be* α*-analytic in the neighborhood of* z_0.

The particular cases (5), (6), and (7) are respectively Taylor's expansion

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,\qquad \tau\geqslant 0, \tag{12}$$

Stirling's expansion

$$f(z)=\sum_{n=0}^{\infty}\frac{\Delta^n f\left(z_0-\frac{n}{2}\right)}{n!}\frac{(z-z_0)\,\Gamma\left(z-z_0+\frac{n}{2}\right)}{\Gamma\left(z-z_0+1-\frac{n}{2}\right)},\qquad \tau<\ln(3+2\sqrt{2}), \tag{13}$$

and (essentially) Newton's expansion

$$f(z)=\sum_{n=0}^{\infty}\Delta^n f(z_0)\binom{z-z_0}{n},\qquad \tau<\ln 2. \tag{14}$$

The associated polynomials $\phi_{n,\alpha}(z)$ are

$$\phi_{n,1/2}(z)=z^n, \tag{15}$$

$$\phi_{n,\infty}(z)=\frac{z\Gamma\left(z+\frac{n}{2}\right)}{\Gamma\left(z+1-\frac{n}{2}\right)}=z\left(z+1-\frac{n}{2}\right)_{n-1}, \tag{16}$$

and

$$\phi_{n,-1/2}(z)=\sum_{r=1}^{n}\frac{n!}{2^{n-r}}\binom{n-1}{r-1}\binom{z}{r}. \tag{17}$$

The expansion (10) can be put in a more general context (Subsection 2.1) where a remainder can be presented with the concept of indicator diagram of f. Our method gives another representation for the remainder associated with the expansion (10). In order to state the result in question, it is useful to introduce the α-integral

$$_{\alpha}\int_{z_0}^{z} f(t)\,dt:=\sum_{k=1}^{\infty}\frac{b_{\alpha}^{(k-1)}(f;z_0)}{k!}\phi_{k,\alpha}(z-z_0) \tag{18}$$

if the series converges. It will become clear that the following identities hold,

$$ {}_{\alpha}\int_{z_0}^{z} b_\alpha(f(t); t)\, dt = f(z) - f(z_0), \tag{19} $$

$$ b_\alpha\left({}_{\alpha}\int_{z_0}^{z} f(t)\, dt; z\right) = f(z), \tag{20} $$

$$ {}_{\alpha}\int_{z_0}^{z} f(t)\, dt = -{}_{\alpha}\int_{z}^{z_0} f(t)\, dt \tag{21} $$

so that

$$ b_\alpha\left({}_{\alpha}\int_{z_0}^{z} f(t)\, dt; z_0\right) = -f(z_0), \tag{22} $$

$$ {}_{\alpha}\int_{z_0}^{z} f(-t)\, dt = -{}_{\alpha}\int_{-z_0}^{-z} f(t)\, dt, \tag{23} $$

$$ {}_{\alpha}\int_{z_0}^{z} f(t+t_0)\, dt = {}_{\alpha}\int_{z_0+t_0}^{z+t_0} f(t)\, dt \tag{24} $$

and the additivity property

$$ {}_{\alpha}\int_{z_0}^{z} f(t)\, dt = {}_{\alpha}\int_{z_0}^{\zeta} f(t)\, dt + {}_{\alpha}\int_{\zeta}^{z} f(t)\, dt. \tag{25} $$

The classical cases $\alpha = \frac{1}{2}$, $\alpha \to \infty$, and $\alpha = -\frac{1}{2}$ are respectively

$$ {}_{1/2}\int_{z_0}^{z} f(t)\, dt = \int_{z_0}^{z} f(t)\, dt, \tag{26} $$

$$ {}_{\infty}\int_{z_0}^{z} f(t)\, dt = \sum_{k=1}^{\infty} \frac{\Delta^{k-1} f\left(z_0 - \dfrac{(k-1)}{2}\right)}{k!} \frac{(z-z_0)\, \Gamma\left(z - z_0 + \dfrac{k}{2}\right)}{\Gamma\left(z - z_0 + 1 - \dfrac{k}{2}\right)} \tag{27} $$

and (to be justified)

$$ {}_{-1/2}\int_{z_0}^{z} f(t)\, dt = \sum_{k=0}^{\infty} \Delta^k f(z_0) \binom{z-z_0}{k+1} + \tfrac{1}{2}\left(f(z) - f(z_0)\right). \tag{28} $$

Formula (28) readily gives

$$_{-1/2}\int_0^N f(t)\,dt = \sum_{k=0}^{N} f(k) - \tfrac{1}{2}\,(f(N)+f(0)) \tag{29}$$

if N is a nonnegative integer.

With the help of the α-integral it becomes possible to give a representation for a remainder which can be considered in (10). The representation is explicit in the sense that

$$\begin{aligned} f(z) = \sum_{n=0}^{N} \frac{b_\alpha^{(n)}(f; z_0)}{n!}\,\phi_{n,\alpha}(z-z_0) + R_{N,\alpha}(z_0, z) \\ + \sum_{n=1}^{N} \sum_{m=2}^{\infty} \sum_{k=1}^{m} \frac{(-1)^k}{(n-k)!}\,c_{k,m}(1,\alpha)\,{}_\alpha\!\int_{z_0}^{z} \phi_{n-k,\alpha}(z-t)\,b_\alpha^{(n-k+m)}(f; t)\,dt, \end{aligned} \tag{30}$$

where

$$R_{N,\alpha}(z_0, z) := \frac{1}{N!}\,{}_\alpha\!\int_{z_0}^{z} \phi_{N,\alpha}(z-t)\,b_\alpha^{(N+1)}(f; t)\,dt \tag{31}$$

and

$$c_{k,m}(1,\alpha) := \frac{1}{k!(m-k)!}\,b_\alpha(\phi_{k,\alpha}(\zeta)\,\phi_{m-k,\alpha}(\zeta); \zeta=0). \tag{32}$$

We now describe the details of the α-calculus.

2. THE α-DIFFERENTIAL CALCULUS

Consider formally the integrals

$$\begin{aligned} I_{N,\alpha} := \frac{1}{N!}\int_1^z B_{N,\alpha}(z-t+1)\,f^{(N+1)}(t)\,dt \\ -\frac{1}{N!}\int_0^z B_{N,\alpha}(z-t)\,f^{(N+1)}(t)\,dt. \end{aligned} \tag{33}$$

The polynomials $B_{n,\alpha}(x)$ satisfy [5] the relations $B'_{n,\alpha}(x)=nB_{n-1,\alpha}(x)$ and $B_{n,\alpha}(1)=(-1)^n B_{n,\alpha}$; using these properties and integrating by parts, we obtain

$$I_{N,\alpha}=\frac{((-1)^N-1)}{N!}B_{N,\alpha}f^{(N)}(z)$$
$$-\frac{1}{N!}B_{N,\alpha}(z)(f^{(N)}(1)-f^{(N)}(0))+I_{N-1,\alpha} \tag{34}$$

and infer the relation

$$I_{N,\alpha}=\sum_{k=0}^{N}\frac{((-1)^k-1)}{k!}B_{k,\alpha}f^{(k)}(z)$$
$$-\sum_{k=0}^{N}\frac{B_{k,\alpha}(z)}{k!}(f^{(k)}(1)-f^{(k)}(0)). \tag{35}$$

Concerning the question of convergence, we prove the following result, from which Theorem A corresponds to $z=0$ and (3) corresponds to $\alpha=\frac{1}{2}$.

THEOREM 2.1. *Let z be a complex number. For each* $f\in B_\tau$ *we have, if* $\tau<2\,|j_1|$,

$$\sum_{k=0}^{\infty}\frac{((-1)^k-1)}{k!}B_{k,\alpha}f^{(k)}(z)=\sum_{k=0}^{\infty}\frac{B_{k,\alpha}(z)}{k!}(f^{(k)}(1)-f^{(k)}(0)). \tag{36}$$

Proof. Given a complex number t we have, by (1),

$$B_{N,\alpha}(t)=\frac{d^N}{dz^N}\left.\frac{e^{(t-1/2)z}}{g_\alpha\left(\frac{iz}{2}\right)}\right|_{z=0}. \tag{37}$$

Using Cauchy's inequalities, we obtain

$$|B_{N,\alpha}(t)|\leqslant\frac{N!}{\rho^N}\max_{|z|=\rho}\left|\frac{e^{(t-1/2)z}}{g_\alpha\left(\frac{iz}{2}\right)}\right|$$
$$\leqslant c_0\frac{N!}{\rho^N} \tag{38}$$

for $\rho < 2\,|j_1|$, where c_0 is independent of N. Now we have

$$I_{N,\alpha} = \frac{1}{N!}\int_1^z B_{N,\alpha}(t)\, f^{(N+1)}(z-t+1)\,dt - \frac{1}{N!}\int_0^z B_{N,\alpha}(t)\, f^{(N+1)}(z-t)\,dt. \tag{39}$$

It follows from (8) and (38) that, for each $\rho < 2\,|j_1|$,

$$|I_{N,\alpha}| \leqslant k_f(\alpha, \tau, z)\left(\frac{\tau}{\rho}\right)^N, \tag{40}$$

where $k_f(\alpha, \tau, z)$ is independent of N. Thus,

$$\lim_{N\to\infty} I_{N,\alpha} = 0 \tag{41}$$

if $\tau < \rho$ and the result follows since ρ can be chosen arbitrarily close to $2\,|j_1|$. ∎

We denote the left-hand member of (36) by $b_\alpha(f; z)$. The sum defining it contains only the derivatives of odd order of f. The particularity of the case $\alpha = \frac{1}{2}$ is that the Bernoulli numbers $B_n =: B_{n,1/2}$ are all equal to zero if $n \geqslant 3$ is odd. It remains then only one term (the one corresponding to $k = 1$) in the left-hand member of (36); that term is $f'(z)$ and we get (3). In that sense, $b_\alpha(f; z)$ is an extension of the ordinary derivative; we may call it the α-derivative (or the Bernoulli–Bessel derivative) of the function f at the point z. In view of Theorem 2.1, $b_\alpha(f; z)$ is certainly defined for all complex numbers z if $f \in B_\tau$, $\tau < 2\,|j_1|$. Also, the α-derivative of a polynomial of degree n is a polynomial of degree $(n-1)$.

It is not difficult to define precisely the α-derivative of order n. We put $b_\alpha^{(0)}(f; z) := f(z)$ and

$$b_\alpha^{(n)}(f; z) := b_\alpha^{(n-1)}(b_\alpha(f; z); z) \tag{42}$$

for $n = 1, 2, 3, \ldots$. We have

$$b_\alpha^{(n)}(f; z) = \sum_{k=0}^{\infty} \frac{d_{k,\alpha}^{(n)}}{k!}\, f^{(k)}(z), \tag{43}$$

where $d_{k,\alpha}^{(1)} := ((-1)^k - 1)\, B_{k,\alpha}$ and

$$d_{k,\alpha}^{(n)} := \sum_{\ell=0}^{k} \binom{k}{\ell} ((-1)^{k-\ell} - 1)\, B_{k-\ell,\alpha}\, d_{\ell,\alpha}^{(n-1)} \tag{44}$$

for $n=2, 3, \ldots$. Also, $d_{0,\alpha}^{(0)} := 1$ and $d_{0,\alpha}^{(n)} := 0$ for $n \geqslant 1$. The sequences $d_{k,\alpha}^{(n)}$ are fundamental quantities in our theory. Using the relation $B_{k,\alpha}(1) - B_{k,\alpha}(0) = ((-1)^k - 1)\, B_{k,\alpha}$ and (1), we see that

$$d_{k,\alpha}^{(1)} = \frac{d^k}{dz^k} \left(\frac{e^{z/2} - e^{-z/2}}{g_\alpha\left(\frac{iz}{2}\right)} \right)^n \Bigg|_{z=0}. \tag{45}$$

By mathematical induction, we then obtain

$$d_{k,\alpha}^{(n)} = \frac{d^k}{dz^k} \left(\frac{e^{z/2} - e^{-z/2}}{g_\alpha\left(\frac{iz}{2}\right)} \right)^n \Bigg|_{z=0}. \tag{46}$$

Note also that, from (43) where $f(z) = z^m$,

$$d_{m,\alpha}^{(n)} = b_\alpha^{(n)}(\zeta^m; \zeta = 0). \tag{47}$$

In particular, for $n \geqslant 1$,

$$d_{k,\alpha}^{(n)} = 0, \qquad 0 \leqslant k < n \tag{48}$$

and, since the function $(e^{z/2} - e^{-z/2})/g_\alpha(iz/2)$ is odd,

$$d_{k,\alpha}^{(n)} = 0 \qquad \text{if} \quad k - n \text{ is odd}, \quad k > n. \tag{49}$$

Some examples of non-zero coefficients are

$$d_{n,\alpha}^{(n)} = n!,$$

$$d_{n+2,\alpha}^{(n)} = \frac{n(n+2)!\,(2\alpha - 1)}{48(\alpha+1)},$$

and

$$d_{n+4,\alpha}^{(n)} = \frac{n(n+4)!\,(2\alpha-1)(-(10n+29) + (15n-18)\,\alpha + (10n-4)\,\alpha^2)}{23040(\alpha+1)^2\,(\alpha+2)}.$$

A more general relation than (46) is

$$\sum_{\ell=0}^{k} \binom{k}{\ell} d_{\ell,\alpha}^{(n)} F^{(k-\ell)}(0) = \frac{d^k}{dz^k} \left(\frac{e^{z/2} - e^{-z/2}}{g_\alpha\left(\frac{iz}{2}\right)} \right)^n F(z) \Bigg|_{z=0}. \tag{50}$$

The choice $F(z)=((e^{z/2}-e^{-z/2})/g_\alpha(iz/2))^m$, in (50), gives the relation

$$d_{k,\alpha}^{(m+n)}=\sum_{\ell=0}^{k}\binom{k}{\ell}d_{\ell,\alpha}^{(m)}d_{k-\ell,\alpha}^{(n)}. \tag{51}$$

Here are some examples. We have

$$b_\alpha(z^m;z)=B_{m,\alpha}(z+1)-B_{m,\alpha}(z), \tag{52}$$

which follows from the definition (4) and the relation [5, p. 309] $B_{m,\alpha}(x+y)=\sum_{k=0}^{m}\binom{m}{k}B_{k,\alpha}(y)\,x^{m-k}$. Also, we have

$$b_\alpha^{(n)}(\phi_{m,\alpha}(z);z)=\frac{m!}{(m-n)!}\,\phi_{m-n,\alpha}(z), \tag{53}$$

for $n=0, 1, ..., m$. In order to prove (53), we consider the α-expansion (10) where $z_0=0$. Applying the operator b_α on both sides, we obtain

$$b_\alpha(f;z)=\sum_{m=1}^{\infty}\frac{b_\alpha^{(m)}(f;0)}{m!}\,b_\alpha(\phi_{m,\alpha}(z);z). \tag{54}$$

On the other hand, the α-expansion (10) applied to $b_\alpha(f;z)$ gives

$$\begin{aligned}b_\alpha(f;z)&=\sum_{m=0}^{\infty}\frac{b_\alpha^{(m+1)}(f;0)}{m!}\,\phi_{m,\alpha}(z)\\&=\sum_{m=1}^{\infty}\frac{b_\alpha^{(m)}(f;0)}{m!}\,m\phi_{m-1,\alpha}(z).\end{aligned} \tag{55}$$

It readily follows from (54) and (55) that

$$b_\alpha(\phi_{m,\alpha}(z);z)=m\phi_{m-1,\alpha}(z), \tag{56}$$

and (53) is simply the direct extension. Another formula, which follows from (43) and (46), is

$$b_\alpha^{(n)}(e^{cz};z)=\left(\frac{e^{c/2}-e^{-c/2}}{g_\alpha(ic/2)}\right)^n e^{cz} \tag{57}$$

for $|c|<2\,|j_1|$.

The α-derivative is not defined in general for functions like $f(z)=\frac{1}{z}$. Formally, we have

$$b_\alpha^{(n)}\left(\frac{1}{z};z\right)=\sum_{k=0}^{\infty}\frac{(-1)^k\,d_{k,\alpha}^{(n)}}{z^{k+1}}, \tag{58}$$

but this series may not converge even for large values of $|z|$. We would want to treat $b_\alpha(\frac{1}{z}; z)$ as a kind of "α-distribution"; we will not go further in that direction except to note that

$$b_\alpha^{(n)}(\delta(z); z) = z b_\alpha^{(n)}\left(\frac{1}{z}; z\right) \delta(z), \tag{59}$$

where δ is the Dirac distribution. For $\alpha = -\frac{1}{2}$ it is possible to extend the definition of the $(-\frac{1}{2})$-derivative so that $b_{-1/2}(\frac{1}{z}; z)$ has a sense for all $z \neq 0$, -1, -2, ... (see Subsection 4.1).

Some properties of the α-derivative follow at once from the definition. The operator b_α is linear and

$$b_\alpha(b_\beta(f; z); z) = b_\beta(b_\alpha(f; z); z). \tag{60}$$

We have also

$$b_\alpha(f(z+\zeta); z = z_0) = b_\alpha(f(z); z = z_0 + \zeta) \tag{61}$$

and

$$b_\alpha(f(-z); z = z_0) = -b_\alpha(f(z); z = -z_0). \tag{62}$$

Other simple properties are already less obvious. For example, no simple expression for $b_\alpha(f(cz); z)$, in terms of the $b_\alpha^{(k)}(f(\zeta); \zeta = cz)$, seems to be easily available, except for $c = 0, \pm 1$. In order to obtain non-trivial properties, we introduce a representation formula.

2.1. *The α-Expansion*

We examine briefly a general situation. See [2, Sect. 9.10] for generalities and [4] for more specific details. Consider the expansion

$$e^{zw} = \sum_{n=0}^{\infty} G_n(w)\, u_n(z), \tag{63}$$

where the $G_n(w)$ are continuous and the convergence is uniform on a contour Γ containing the indicator diagram of $f(z)$. There is a representation of the form

$$f(z) = \sum_{n=0}^{\infty} T_n(f)\, u_n(z), \tag{64}$$

where

$$T_n(f) := \frac{1}{2\pi i} \int_\Gamma \phi(w)\, G_n(w)\, dw. \tag{65}$$

The function $\phi(w)$ is the Borel transform of f. A case of particular interest is $G_n(w) = (\zeta(w))^n$ where $\zeta(w)$ is analytic and univalent in the neighborhood of the origin, with $\zeta(0) = 0$. Then we have

$$e^{zw} = \sum_{n=0}^{\infty} \zeta^n u_n(z), \tag{66}$$

where

$$u_n(z) = \frac{1}{n!} \frac{d^n}{d\zeta^n} e^{zw(\zeta)} \bigg|_{\zeta=0}. \tag{67}$$

The classical expansions are

(A) Taylor Series. We take $\zeta(w) = w$ so that $w(\zeta) = \zeta$, $u_n(z) = z^n/n!$, and $T_n(f) = f^{(n)}(0)$.

(B) Stirling Series. We take $\zeta(w) = e^{w/2} - e^{-w/2}$ so that $w(\zeta) = 2 \ln((\zeta + \sqrt{\zeta^2 + 4})/2)$, $u_n(z) = z\Gamma(z + n/2)/n!\, \Gamma(z + 1 - n/2)$, and $T_n(f) = \Delta^n f(-\frac{n}{2})$.

(C) Newton Series. We take $\zeta(w) = e^w - 1$ so that $w(\zeta) = \ln(1 + \zeta)$, $u_n(z) = \binom{z}{n}$ and $T_n(f) = \Delta^n f(0)$.

The α-expansion (10) corresponds to the choice

$$\zeta(w) = \frac{e^{w/2} - e^{-w/2}}{g_\alpha\left(\dfrac{iw}{2}\right)}, \tag{68}$$

with $T_n(f) = b_\alpha^{(n)}(f(z); z = 0)$. This follows from (65), (57), and the Pólya representation [2, p. 74]

$$f(z) = \frac{1}{2\pi i} \int_\Gamma e^{wz} \phi(w)\, dw. \tag{69}$$

It is also a consequence of (65), (43), and the relation (see (46))

$$(\zeta(w))^n = \sum_{k=0}^{\infty} \frac{d_{k,\alpha}^{(n)}}{k!} w^k. \tag{70}$$

Remark 2.1. It is not without interest to observe here that (56) can be obtained from (67) and (57):

$$b_\alpha(\phi_{m,\alpha}(z); z) = b_\alpha\left(\frac{d^m e^{zw}}{d\zeta^m}\bigg|_{\zeta=0}; z\right)$$

$$= \frac{d^m}{d\zeta^m}\left(\frac{e^{w/2} - e^{-w/2}}{g_\alpha\left(\frac{iw}{2}\right)}\right) e^{zw}\Bigg|_{\zeta=0}$$

$$= \frac{d^m}{d\zeta^m}\zeta e^{zw}\bigg|_{\zeta=0} \qquad \text{by (68)}$$

$$= m\,\frac{d^{m-1}e^{zw}}{d\zeta^{m-1}}\bigg|_{\zeta=0}$$

$$= m\phi_{m-1,\alpha}(z).$$

Before we present our second theorem, we observe that the condition $b_\alpha(f; z) \equiv 0$, where f is α-analytic (i.e., expressible in the form (10)), implies that $f(z)$ is a constant. Indeed, if $b_\alpha(f(z); z = z_0) = 0$ for all z_0 then $b_\alpha^{(n)}(f(z); z = z_0) \equiv 0$, $n \geqslant 1$, and so

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{b_\alpha^{(n)}(f; z_0)}{n!}\phi_{n,\alpha}(z - z_0)$$

$$= f(z_0).$$

Looking now at (57), we see that

$$b_\alpha(e^{2\ell\pi i z}; z) \equiv 0 \tag{71}$$

if ℓ is an integer such that $|\ell| < |j_1|/\pi$.

Remark 2.2. For $\alpha = \frac{1}{2}$ we have $g_{1/2}(ic/2) = (e^{c/2} - e^{-c/2})/c$ so that the factor $(e^{c/2} - e^{-c/2})$ cancels out in (57). We have $j_1(\frac{1}{2}) = \pi$ and the only admissible integer in (71) is $\ell = 0$.

The relations (71) means that the function $f(z) = e^{cz}$ cannot be α-analytic for $|c| \geqslant 2\pi$ if $|j_1| > \pi$, which is certainly true for $\alpha > \frac{1}{2}$ since [10, p. 508] the positive zeros of $J_\alpha(z)$ increase as α is increased. This proves the last assertion of the following result.

THEOREM 2.2. *Let z_0 and z be complex numbers. There exists a positive constant $\tau(\alpha)$ such that the expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{b_\alpha^{(n)}(f; z_0)}{n!} \phi_{n,\alpha}(z - z_0) \tag{72}$$

holds for all $f \in B_\tau$ if $\tau < \tau(\alpha)$. Moreover, we have $\tau(\alpha) < 2\pi$ if $\alpha > \frac{1}{2}$.

The proof of Theorem 2.2 will be a direct consequence of the next two lemmas.

LEMMA 2.1. *Let z be a complex number. There exists a positive constant $\rho_1 = \rho_1(\alpha)$ such that*

$$|\phi_{n,\alpha}(z)| \leqslant c_1 \frac{n!}{\rho_1^n}, \tag{73}$$

where c_1 is independent of n.

Proof. The function $\zeta(w)$ defined by (68) is analytic for $|w| < 2\,|j_1|$ and $\zeta(0) = 0$. By the Lagrange–Bürmann Theorem [8, Sect. 2.4] it has an analytic inverse $w = w(\zeta)$ in a neighborhood of $\zeta = 0$. Thus, using (67) and Cauchy's inequalities, we obtain

$$\begin{aligned} |\phi_{n,\alpha}(z)| &= n!\; |u_n(z)| \\ &= \left| \frac{d^n}{d\zeta^n} e^{zw(\zeta)} \Big|_{\zeta=0} \right| \\ &\leqslant \frac{n!}{\rho^n} \max_{|\zeta| = \rho} |e^{zw(\zeta)}|, \end{aligned}$$

if ρ is sufficiently small. ∎

Remark 2.3. (a) Since $\phi_{n,\alpha}(0) = 0$, $n \geqslant 1$, we deduce at once from the above inequality and Schwarz Lemma that

$$|\phi_{n,\alpha}(z)| \leqslant c^* \frac{n!}{\rho_1^n} |z| \tag{74}$$

for $|z| \leqslant 1$, where c^* is independent of n and z.

(b) Formula (17) readily implies that

$$|\phi_{n,-1/2}(z)| \leqslant n!\, (\tfrac{3}{2})^{n-1}, \qquad n \geqslant 1, \tag{75}$$

for $|z| \leqslant 1$, with $|\phi_{n,-1/2}(1)| = n!/2^{n-1}$. The bound given in (73) is thus essentially best possible for arbitrary α. We note also that, in view of (16) and Stirling's formula,

$$\phi_{n,\infty}(z) \sim \frac{zn! \sin\left(\frac{n\pi}{2} - \pi z\right)}{\sqrt{2\pi}\, n^{3/2} 2^{n-2}} \tag{76}$$

as $n \to \infty$.

Lemma 2.2. *Let ρ_2 be a positive number such that $\rho_2 < 2\,|j_1|$. We have*

$$|d_{k,\alpha}^{(n)}| \leqslant c_2^n \frac{k!}{\rho_2^k}, \tag{77}$$

where c_2 is independent of n and k.

Proof. The inequality (77) follows from (46) with another application of Cauchy's inequalities. ∎

Remark 2.4. A straightforward consequence of (43), (8), and (77) is an inequality of the form

$$|b_\alpha^{(n)}(f; z)| \leqslant M^* c_3^n e^{\tau\,|y|}, \tag{78}$$

which holds for each $f \in B_\tau$ if $\tau < 2\,|j_1|$. In particular, the class B_τ is closed under α-differentiation whenever $\tau < 2\,|j_1|$.

Proof of Theorem 2.2. The formal proof runs as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{b_\alpha^{(n)}(f; z_0)}{n!} \phi_{n,\alpha}(z - z_0) &= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{d_{k,\alpha}^{(n)}}{k!\, n!} f^{(k)}(z_0)\, \phi_{n,\alpha}(z - z_0) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{d_{n,\alpha}^{(k)}}{n!\, k!} f^{(n)}(z_0)\, \phi_{k,\alpha}(z - z_0) \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \qquad \text{by (11) and (47),} \\
&= f(z).
\end{aligned}$$

Thus, we need only to justify the change in the order of summation. The series in consideration are absolutely convergent since, by Lemmas 2.1 and 2.2, they are bounded by geometrical series which converge for all z if τ is sufficiently small. ∎

2.2. *The Product Formula*

Leibniz's formula

$$(f(z)\, g(z))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z) \tag{79}$$

is one of the most classical result of differential calculus. It can be proved easily by mathematical induction. In order to obtain an extension of this formula, we choose another approach. We write

$$\begin{aligned} f(z)\, g(z) &= \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \right) \left(\sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z-z_0)^k \right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{f^{(k)}(z_0)}{k!} \frac{g^{(m-k)}(z_0)}{(m-k)!} (z-z_0)^m, \end{aligned}$$

whence

$$\begin{aligned} \left. \frac{d^n f(z)\, g(z)}{dz^n} \right|_{z=z_0} &= \lim_{z \to z_0} \sum_{m=n}^{\infty} \sum_{k=0}^{m} \frac{f^{(k)}(z_0)}{k!} \frac{g^{(m-k)}(z_0)}{(m-k)!} \cdot \frac{m!}{(m-n)!} (z-z_0)^{m-n} \\ &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0)\, g^{(n-k)}(z_0). \end{aligned}$$

This simple method leads us to the following extension of (79).

THEOREM 2.3. *For $n = 0, 1, 2, \ldots$, we have the relation*

$$b_\alpha^{(n)}(f(z)\, g(z); z) = \sum_{m=n}^{\infty} \sum_{k=0}^{m} c_{k,m}(n, \alpha)\, b_\alpha^{(k)}(f(z); z)\, b_\alpha^{(m-k)}(g(z); z), \tag{80}$$

where

$$c_{k,m}(n, \alpha) := \frac{1}{k!\,(m-k)!} b_\alpha^{(n)}(\phi_{k,\alpha}(\zeta)\, \phi_{m-k,\alpha}(\zeta); \zeta = 0). \tag{81}$$

Some simple calculations show that

$$c_{k,m}(n, \alpha) = 0 \qquad \text{if} \quad (m-n) \text{ is odd}, \tag{82}$$

and

$$c_{k,m}(m, \alpha) = \binom{m}{k}, \qquad 0 \leqslant k \leqslant m. \tag{83}$$

Proof of Theorem 2.3. Using the α-expansion (10), we may write

$$f(z)\,g(z)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{b_\alpha^{(k)}(f;z_0)}{k!}\frac{b_\alpha^{(m-k)}(g;z_0)}{(m-k)!}\phi_{k,\alpha}(z-z_0)\,\phi_{m-k,\alpha}(z-z_0),$$

whence

$$b_\alpha^{(n)}(f(z)\,g(z);z)=\sum_{m=n}^{\infty}\sum_{k=0}^{m}\frac{b_\alpha^{(k)}(f;z_0)}{k!}\frac{b_\alpha^{(m-k)}(g;z_0)}{(m-k)!}\cdot b_\alpha^{(n)}(\phi_{k,\alpha}(z-z_0)\,\phi_{m-k,\alpha}(z-z_0);z),$$

from which (80) follows with the help of (61). ∎

Applying (80) to $f(z)=e^{az}$ and $g(z)=e^{bz}$, we obtain the following generalization of the binomial formula

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}. \tag{84}$$

COROLLARY 2.1. *Let n be a positive integer. We have the relation*

$$\left(\frac{e^{(a+b)/2}-e^{-(a+b)/2}}{g_\alpha\left(i\,\dfrac{(a+b)}{2}\right)}\right)^n=\sum_{m=n}^{\infty}\sum_{k=0}^{m}c_{k,m}(n,\alpha)\left(\frac{e^{a/2}-e^{-a/2}}{g_\alpha\left(i\,\dfrac{a}{2}\right)}\right)^k\left(\frac{e^{b/2}-e^{-b/2}}{g_\alpha\left(i\,\dfrac{b}{2}\right)}\right)^{m-k} \tag{85}$$

if the series converge.

Formula (85) can also be seen as an addition formula for $g_\alpha(z)$. The relation (84) corresponds to $\alpha=\frac{1}{2}$. For $\alpha\to\infty$, it follows from (124), the polynomial inequality $|\Delta^nP(z)|\leqslant(m!/(m-n)!)\max_{|\zeta|=|z|+n}|P(\zeta)|$ and (73) that the series in (85) are absolutely convergent for $|a|$, $|b|$ sufficiently small. When $\alpha=-\frac{1}{2}$, we may also obtain estimates for $|c_{k,m}(n,-\frac{1}{2})|$ and infer the absolute convergence at least for small $|a|$, $|b|$.

Remark 2.5. In the context of the general situation encountered at the beginning of Subsection 2.1, it is possible to give a more algebraic extension of (84). It follows indeed from (67) that (79) that

$$u_n(a+b)=\sum_{k=0}^{n}u_k(a)\,u_{n-k}(b). \tag{86}$$

The limiting case $\alpha \to \infty$ of (86) gives, in view of (16) and (68), the gamma identity

$$\frac{(a+b)\,\Gamma\left(a+b+\frac{n}{2}\right)}{\Gamma\left(a+b+1-\frac{n}{2}\right)}=\sum_{k=0}^{n}\binom{n}{k}\frac{a\Gamma\left(a+\frac{k}{2}\right)}{\Gamma\left(a+1-\frac{k}{2}\right)}\cdot\frac{b\Gamma\left(b+\frac{(n-k)}{2}\right)}{\Gamma\left(b+1-\frac{(n-k)}{2}\right)}. \tag{87}$$

We can also deduce from Theorem 2.3 another (see (51)) addition formula for the quantities $d_{k,\alpha}^{(n)}$.

COROLLARY 2.2. *The coefficients* $d_{k,\alpha}^{(n)}$ *defined by* (44) *satisfy the relation*

$$d_{j+k,\alpha}^{(n)}=\sum_{m=n}^{\infty}\sum_{\ell=0}^{m}c_{\ell,m}(n,\alpha)\,d_{j,\alpha}^{(\ell)}d_{k,\alpha}^{(m-\ell)}. \tag{88}$$

Proof. Using (43) and (79), we obtain

$$\begin{aligned}b_{\alpha}^{(n)}(f(z)\,g(z);z)&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{d_{k,\alpha}^{(n)}}{k!}f^{(j)}(z)\,g^{(k-j)}(z)\\&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}d_{j+k,\alpha}^{(n)}\frac{f^{(k)}(z)}{k!}\frac{g^{(j)}(z)}{j!}.\end{aligned} \tag{89}$$

On the other hand, Theorem 2.3 and (43) give

$$b_{\alpha}^{(n)}(f(z)\,g(z);z)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}c_{\ell,m}(n,\alpha)\,d_{j,\alpha}^{(\ell)}d_{k,\alpha}^{(m-\ell)}\frac{f^{(j)}(z)}{j!}\frac{g^{(k)}(z)}{k!}, \tag{90}$$

and the result follows by comparison of (89) and (90). ∎

The particular case $f(z)=z$, in (80), readily gives the

COROLLARY 2.3. *We have the relation*

$$b_{\alpha}^{(n)}(zg(z);z)=zb_{\alpha}^{(n)}(g(z);z)+\sum_{m=n}^{\infty}c_{1,m}(n,\alpha)\,b_{\alpha}^{(m-1)}(g(z);z). \tag{91}$$

2.3. *The Composition Formula*

A general formula for the expansion of $(f(g(z)))^{(n)}$ is that of Faa Di Bruno [7, p. 34], namely

$$(f(g(z)))^{(n)} = \sum_{k=1}^{n} \sum_{\pi(n,k)} c(r_1, ..., r_n) \prod_{j=1}^{n} (g^{(j)}(z))^{r_j} \cdot f^{(k)}(g(z)), \tag{92}$$

where $c(r_1, ..., r_n) := n!/(r_1! \cdots r_n!(1!)^{r_1} \cdots (n!)^{r_n})$ and $\pi(n, k)$ means that the summation is over all the nonnegative integers $r_1, ..., r_n$ such that $r_1 + 2r_2 + \cdots + nr_n = n$ and $r_1 + r_2 \cdots + r_n = k$. In order to extend (92), we consider the α-expansion

$$f(g(z)) = \sum_{k=0}^{\infty} \frac{1}{k!} b_\alpha^{(k)}(f(\zeta); \zeta = g(z_0))\, \phi_{k,\alpha}(g(z) - g(z_0)). \tag{93}$$

Applying the operator $b_\alpha^{(n)}(\ ; z)$ on both sides of (93), we obtain

$$b_\alpha^{(n)}(f(g(z)); z = z_0) = \lim_{z \to z_0} \sum_{k=0}^{\infty} \frac{1}{k!} b_\alpha^{(k)}(f(\zeta); \zeta = g(z_0)) \cdot b_\alpha^{(n)}(\phi_{k,\alpha}(g(z) - g(z_0)); z). \tag{94}$$

We have thus the following result.

THEOREM 2.4. *We have the composition formula*

$$b_\alpha^{(n)}(f(g(z)); z) = \sum_{k=0}^{\infty} \frac{1}{k!} b_\alpha^{(n)}(\phi_{k,\alpha}(g(\zeta) - g(z)); \zeta = z) \cdot b_\alpha^{(k)}(f(\zeta); \zeta = g(z)). \tag{95}$$

Let us show that (92) is a special case of Theorem 2.4. For $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} b_\alpha^{(n)}(\phi_{k,\alpha}(g(\zeta) - g(z)); \zeta = z) &= \frac{d^n}{d\zeta^n}(g(\zeta) - g(z))^k \Big|_{\zeta = z} \\ &= \lim_{\zeta \to z} \frac{d^n}{d\zeta^n} \sum_{v_1=1}^{\infty} \cdots \sum_{v_k=1}^{\infty} \prod_{j=1}^{k} \left(\frac{g^{(v_j)}(z)}{v_j!}\right) (\zeta - z)^{v_1 + \cdots + v_k} \\ &= \sum_{v_1 + \cdots + v_k = n} \frac{n!}{v_1! \cdots v_k!} \prod_{j=1}^{k} (g^{(v_i)}(z)), \end{aligned}$$

and we obtain a formula which is known to be equivalent to (92).

Remark 2.6. We can also expand $b_\alpha^{(n)}(f(g(z)); z)$ starting with (43) and using (92). It results an expression which gives, when compared with the right-hand member of (95), the formula

$$\sum_{k=1}^{\ell} \frac{d_{\ell,\alpha}^{(k)}}{k!\,\ell!}\, b_\alpha^{(n)}(\phi_{k,\alpha}(g(\zeta)-g(z)); \zeta=z)$$

$$= \sum_{k=\ell}^{\infty} \sum_{\pi(k,\ell)} \frac{d_{k,\alpha}^{(n)}}{k!}\, c(r_1, ..., r_k) \prod_{j=1}^{k} (g^{(j)}(z))^{r_j}, \tag{96}$$

for $\ell = 1, 2, 3, ...$.

The expression $b_\alpha^{(n)}(\phi_{k,\alpha}(g(\zeta)-g(z)); \zeta=z)$, in (95), cannot be simplified easily in general. It remains however some cases of particular interest. Consider the example $g(z)=e^z$. The formula

$$(f(e^z))^{(n)} = \sum_{k=1}^{n} S(n,k)\, e^{kz} f^{(k)}(e^z) \tag{97}$$

may be used to introduce the so-called Stirling's numbers of the second kind $S(n,k)$, $1 \leqslant k \leqslant n$. We have $S(n,1)=S(n,n)=1$ and $S(n,k) = kS(n-1,k)+S(n-1,k-1)$ for $1<k<n$. Using (43) and (97), we can write

$$b_\alpha^{(n)}(f(e^z); z) = \sum_{k=1}^{\infty} \sum_{r=k}^{\infty} \frac{d_{r,\alpha}^{(n)}}{r!}\, S(r,k)\, e^{kz} f^{(k)}(e^z), \tag{98}$$

where

$$\sum_{r=k}^{\infty} \frac{d_{r,\alpha}^{(n)}}{r!}\, S(r,k) =: S_\alpha(n,k) \tag{99}$$

may be called the α-Stirling's numbers of the second kind. For instances, we have $S_{1/2}(n,k)=S(n,k)$ and some elementary combinatorial identities lead us to

$$k!\, S_\infty(n,k) = \sum_{\nu=0}^{k-1} (-1)^\nu \binom{k}{\nu} (e^{(k-\nu)/2} - e^{-(k-\nu)/2})^n. \tag{100}$$

Another particular value is $S_{-1/2}(n, 1) = (\frac{2(e-1)}{e+1})^n$, $n \geqslant 1$. The α-Stirling's numbers of the first kind may be defined by the relation

$$b_\alpha^{(n)}(f(\ln z); z) = \sum_{k=1}^{\infty} s_\alpha(n, k) \frac{f^{(k)}(\ln z)}{z^n}, \tag{101}$$

with

$$s_\alpha(n, k) := \sum_{r=k}^{\infty} \frac{(-1)^{r+k}}{r!} d_{r,\alpha}^{(n)} s(r, k). \tag{102}$$

The numbers $s(n, k) = s_{1/2}(n, k)$, $1 \leqslant k \leqslant n$, are the usual Stirling's numbers of the first kind, which can be defined by the recurrence relation $s(n, n) = 1$, $s(n, 1) = (n-1)!$ and $s(n, k) = (n-1)\, s(n-1, k) + s(n-1, k-1)$ for $1 < k < n$.

3. THE α-INTEGRAL CALCULUS

The way chosed to introduce the α-integral by (18), namely

$$\begin{aligned} {}_\alpha\!\int_{z_0}^{z} f(t)\, dt = f(z_0)(z - z_0) + \tfrac{1}{2}\, b_\alpha(f; z_0)(z - z_0)^2 \\ + \tfrac{1}{6}\, b_\alpha^{(2)}(f; z_0)\, \phi_{3,\alpha}(z - z_0) + \cdots, \end{aligned} \tag{103}$$

is very natural and properties (19), (20), (23), (24) follow easily from this definition. Some examples of calculations are

$${}_\alpha\!\int_{z_0}^{z} \phi_{n,\alpha}(t)\, dt = \frac{\phi_{n+1,\alpha}(z) - \phi_{n+1,\alpha}(z_0)}{n+1}, \tag{104}$$

$${}_\alpha\!\int_{z_0}^{z} t^n\, dt = \sum_{k=1}^{n+1} \frac{d_{n,\alpha}^{(k-1)}}{k!} (\phi_{k,\alpha}(z) - \phi_{k,\alpha}(z_0)) \tag{105}$$

and

$${}_\alpha\!\int_{z_0}^{z} e^{ct}\, dt = \frac{g_\alpha\left(i\dfrac{c}{2}\right)}{(e^{c/2} - e^{-c/2})} \cdot (e^{cz} - e^{cz_0}) \tag{106}$$

if $|c|$ is sufficiently small.

Proof of (21). Using the α-expansion and (104), we have

$$
\begin{aligned}
{}_\alpha\int_{z_0}^{z} f(t)\,dt &= {}_\alpha\int_{z_0}^{z} \sum_{k=0}^{\infty} \frac{1}{k!}\, b_\alpha^{(k)}(f;0)\, \phi_{k,\alpha}(t)\,dt \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\, b_\alpha^{(k)}(f;0) \cdot \frac{\phi_{k+1,\alpha}(z) - \phi_{k+1,\alpha}(z_0)}{k+1} \\
&= -\sum_{k=0}^{\infty} \frac{1}{k!}\, b_\alpha^{(k)}(f;0) \cdot \frac{\phi_{k+1,\alpha}(z_0) - \phi_{k+1,\alpha}(z)}{k+1} \\
&= -{}_\alpha\int_{z}^{z_0} f(t)\,dt. \quad \blacksquare
\end{aligned}
$$

We present now two proofs of the additivity property (25).

Proof (*First Proof of* (25)). We use the idea of the preceding proof:

$$
\begin{aligned}
{}_\alpha\int_{z_0}^{z} f(t)\,dt &= {}_\alpha\int_{z_0}^{z} \sum_{k=0}^{\infty} \frac{b_\alpha^{(k)}(f;\zeta)}{k!}\, \phi_{k,\alpha}(t-\zeta)\,dt \\
&= \sum_{k=0}^{\infty} \frac{b_\alpha^{(k)}(f;\zeta)}{k!} \cdot \frac{\phi_{k+1,\alpha}(z-\zeta) - \phi_{k+1,\alpha}(z_0-\zeta)}{k+1} \\
&= \sum_{k=1}^{\infty} \frac{b_\alpha^{(k-1)}(f;\zeta)}{k!}\, \phi_{k,\alpha}(z-\zeta) \\
&\quad - \sum_{k=1}^{\infty} \frac{b_\alpha^{(k-1)}(f;\zeta)}{k!}\, \phi_{k,\alpha}(z_0-\zeta) \\
&= {}_\alpha\int_{\zeta}^{z} f(t)\,dt - {}_\alpha\int_{\zeta}^{z_0} f(t)\,dt \\
&= {}_\alpha\int_{z_0}^{\zeta} f(t)\,dt + {}_\alpha\int_{\zeta}^{z} f(t)\,dt. \quad \blacksquare
\end{aligned}
$$

Proof (*Second Proof of* (25)). Recall that an α-analytic function F such that $b_\alpha(F(z);z) \equiv 0$ must be a constant. Let

$$
F(z) := {}_\alpha\int_{z_0}^{z} f(t)\,dt - {}_\alpha\int_{z_0}^{\zeta} f(t)\,dt - {}_\alpha\int_{\zeta}^{z} f(t)\,dt.
$$

Using (20), we have $b_\alpha(F(z);z) = f(z) - f(z) = 0$ for all z. Thus, $F(z) \equiv F(z_0) = 0$ by (21) and the result follows. ∎

We look now more closely at formula (30). We will deduce it from a general version of the integration by parts formula.

THEOREM 3.1. *The following formula holds:*

$$
\begin{aligned}
{}_{\alpha}\!\int_{z_0}^{z} f(t)\, b_\alpha(g(t); t)\, dt
&= f(z)\, g(z) - f(z_0)\, g(z_0) - {}_{\alpha}\!\int_{z_0}^{z} b_\alpha(f(t); t)\, g(t)\, dt \\
&\quad - \sum_{m=2}^{\infty} \sum_{k=1}^{m} c_{k,m}(1, \alpha)\, {}_{\alpha}\!\int_{z_0}^{z} b_\alpha^{(k)}(f(t); t)\, b_\alpha^{(m-k)}(g(t); t)\, dt. \qquad (107)
\end{aligned}
$$

Proof. For $n=1$, the product formula (80) can be written as

$$
\begin{aligned}
b_\alpha(f(t)\, g(t); t) &= f(t)\, b_\alpha(g(t); t) + b_\alpha(f(t); t)\, g(t) \\
&\quad + \sum_{m=2}^{\infty} \sum_{k=1}^{m} c_{k,m}(1, \alpha)\, b_\alpha^{(k)}(f(t); t)\, b_\alpha^{(m-k)}(g(t); t). \qquad (108)
\end{aligned}
$$

Formula (107) follows from (19) when an α-integration is performed on both sides of (108). ▮

The first α-integral appearing in the right-hand member of formula (107) is the case $m=1$ in the double summation. Written in the way chosed, we see at once that (107) is a direct generalization of the classical integration by parts formula since $c_{k,m}(1, \frac{1}{2}) = 0$ for $m \geqslant 2$, $1 \leqslant k \leqslant m$. It has also the appropriate form for an immediate application.

A limited version of the general expansion (64) is obtainable from (65) when $G_n(w) = (\zeta(w))^n$, with $|\zeta(w)| < 1$ on Γ. Then we have

$$
f(z) = \sum_{m=0}^{N} T_n(f)\, u_n(z) + \int_\Gamma \phi(w)\, \frac{(\zeta(w))^{N+1}}{1 - \zeta(w)}\, dw. \qquad (109)
$$

In the case (68), the relation (30) is a new form for the remainder. We replace, in (107), $f(t)$ by $\frac{1}{n!}\,\phi_{n,\alpha}(z-t)$ and $g(t)$ by $b_\alpha^{(n)}(f(t); t)$. The application of Theorem 3.1 then gives

$$
\begin{aligned}
R_{n,\alpha}(z_0, z) &= -\frac{1}{n!}\, \phi_{n,\alpha}(z - z_0)\, b_\alpha^{(n)}(f; z_0) + R_{n-1,\alpha}(z_0, z) \\
&\quad - \sum_{m=2}^{\infty} \sum_{k=1}^{m} \frac{(-1)^k}{(n-k)!}\, c_{k,m}(1, \alpha) \\
&\quad \times {}_{\alpha}\!\int_{z_0}^{z} \phi_{n-k,\alpha}(z - t)\, b_\alpha^{(n-k+m)}(f(t); t)\, dt, \qquad (110)
\end{aligned}
$$

where $R_{n,\alpha}(z_0, z)$ is defined by (31). The relation (30) is obtained by simplification of the telescoping series $\sum_{n=1}^{N} (R_{n,\alpha}(z_0, z) - R_{n-1,\alpha}(z_0, z))$.

Remark 3.1. It is clear that (30) holds for any polynomial. For $f(z) = z^j$, $0 \leqslant j \leqslant N$, we obtain the relation

$$\sum_{n=1}^{N} \sum_{m=2}^{\infty} \sum_{k=1}^{m} \frac{(-1)^k}{(n-k)!} c_{k,m}(1, \alpha) \,{}_{\alpha}\!\int_{z_0}^{z} \phi_{n-k,\alpha}(z-t)\, b_{\alpha}^{(n-k+m)}(t^j; t)\, dt \equiv 0. \tag{111}$$

4. OTHER RESULTS

4.1. *The case* $\alpha = -\frac{1}{2}$

The relation [7, p. 313] $E_n(0) = \frac{2}{(n+1)} (1 - 2^{n+1}) B_{n+1}(0)$, relating the values of the Bernoulli and Euler polynomials at $x = 0$, gives at once

$$b_{-1/2}(f(z); z) = 4 \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)}{(2k)!} B_{2k} f^{(2k-1)}(z). \tag{112}$$

A particularity of the case $\alpha = -\frac{1}{2}$ is the pair of relations

$$b_{-1/2}^{(n)}(f(z); z) = \sum_{m=n}^{\infty} \frac{(-1)^{m-n}}{2^{m-n}} \binom{m-1}{n-1} \Delta^m f(z), \tag{113}$$

$$\Delta^n f(z) = \sum_{m=n}^{\infty} \frac{1}{2^{m-n}} \binom{m-1}{n-1} b_{-1/2}^{(m)}(f(z); z). \tag{114}$$

That (113) and (114) are equivalent is an immediate consequence of the combinatorial pair

$$a_n = \sum_{m=n}^{\infty} \binom{m-1}{n-1} b_m, \tag{115}$$

$$b_n = \sum_{m=n}^{\infty} (-1)^{m-n} \binom{m-1}{n-1} a_m. \tag{116}$$

Proof of Formula (114)). We may put the $(-\frac{1}{2})$-expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} b_{-1/2}^{(n)}(f; z_0)\, \phi_{n,-1/2}(z - z_0) \tag{117}$$

in the context of (64) where we take $\zeta(w)=2((e^{w/2}-e^{-w/2})/(e^{w/2}+e^{-w/2}))$, so that

$$\phi_{n,\,-1/2}(z)=n!\,u_n(z)=\frac{d^n}{d\zeta^n}\left(\frac{2+\zeta}{2-\zeta}\right)^z\Bigg|_{\zeta=0}=\sum_{r=1}^{n}\frac{n!}{2^{n-r}}\binom{n-1}{r-1}\binom{z}{r},$$

whence

$$\Delta^m\phi_{n,\,-1/2}(z)=\begin{cases}\displaystyle\sum_{r=m}^{n}\frac{n!\binom{n-1}{r-1}\Gamma(z+1)}{(r-m)!\,2^{n-r}\,\Gamma(z+m+1-r)} & \text{if}\quad m\leqslant n,\\ 0 & \text{if}\quad m>n,\end{cases}$$

with, in particular,

$$\Delta^m\phi_{n,\,-1/2}(0)=\begin{cases}\dfrac{n!}{2^{n-m}}\dbinom{n-1}{m-1} & \text{if}\quad m\leqslant n\\ 0 & \text{if}\quad m>n.\end{cases}\tag{118}$$

It follows from (117) and (118) that

$$\begin{aligned}\Delta^m f(z_0)&=\lim_{z\to z_0}\sum_{n=m}^{\infty}\frac{1}{n!}\,b^{(n)}_{-1/2}(f;z_0)\,\Delta^m\phi_{n,\,-1/2}(z-z_0)\\&=\sum_{n=m}^{\infty}\frac{1}{2^{n-m}}\,b^{(n)}_{-1/2}(f;z_0)\binom{n-1}{m-1}.\quad\blacksquare\end{aligned}$$

Newton's expansion (14) follows from (113) since, by (117),

$$\begin{aligned}f(z)&=f(z_0)+\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{(-1)^{m-n}\binom{m-1}{n-1}}{2^{m-n}\,n!}\,\Delta^m f(z_0)\,\phi_{n,\,-1/2}(z-z_0)\\&=f(z_0)+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{n-m}\binom{n-1}{m-1}}{2^{n-m}\,m!}\,\phi_{m,\,-1/2}(z-z_0)\,\Delta^n f(z_0)\\&=f(z_0)+\sum_{n=1}^{\infty}\Delta^n f(z_0)\binom{z-z_0}{n}\\&=\sum_{n=0}^{\infty}\Delta^n f(z_0)\binom{z-z_0}{n}.\end{aligned}$$

In the above reasoning, the identity

$$\sum_{m=1}^{n} \frac{(-1)^{m-n}\binom{n-1}{m-1}}{2^{n-m}\, m!}\, \phi_{m,\,-1/2}(z-z_0) = \binom{z-z_0}{n} \tag{119}$$

has been used. An example of calculation using (113) is

$$b_{-1/2}\left(\binom{z}{k+1} + \frac{1}{2}\binom{z}{k};\, z\right) = \binom{z}{k} \tag{120}$$

for $k = 0, 1, 2, \ldots$. Formula (113) permits to give a sense at $b_{-1/2}(\frac{1}{z};\, z)$. We have indeed $\Delta^m(\frac{1}{z}) = (-1)^m\, m!/(z(z+1)\cdots(z+m)) = \sum_{j=0}^{m} ((-1)^{j+m}\binom{m}{j}/(z+j))$, which gives

$$b_{-1/2}\left(\frac{1}{z};\, z\right) = -\frac{2}{z} - 4\sum_{m=1}^{\infty} \frac{(-1)^m}{z+m}, \tag{121}$$

for $z \neq 0, -1, -2, \ldots$.

At this point, several approaches are available to justify (28). One of them uses the same idea as in the proof of Newton's expansion using (119), except that the identity

$$\sum_{m=1}^{n} \frac{(-1)^{m-n}\binom{n-1}{m-1}}{2^{n-m}(m+1)!}\, \phi_{m+1,\,-1/2}(z-z_0) = \binom{z-z_0}{n} \frac{(2(z-z_0)-(n-1))}{2(n+1)} \tag{122}$$

must be used. We choose the following method. Let

$$G(z) := \sum_{k=0}^{\infty} \Delta^k f(z_0)\binom{z-z_0}{k+1} + \frac{1}{2}(f(z) - f(z_0)).$$

Using the expansion (14), we can write

$$G(z) = f(z_0)(z-z_0) + \sum_{k=1}^{\infty} \Delta^k f(z_0)\left(\binom{z-z_0}{k+1} + \frac{1}{2}\binom{z-z_0}{k}\right),$$

whence, in view of (120),

$$b_{-1/2}(G(z); z) = f(z_0) + \sum_{k=1}^{\infty} \Delta^k f(z_0) \binom{z-z_0}{k}$$
$$= \sum_{k=0}^{\infty} \Delta^k f(z_0) \binom{z-z_0}{k}$$
$$= f(z) \qquad \text{by (14).}$$

It follows that

$$G(z) = {}_{-1/2}\int_{z_0}^{z} f(t)\, dt + c,$$

with $c = G(z_0) = 0$. This proves formula (28). A direct calculation using (28) is

$${}_{-1/2}\int_{z_0}^{z} \frac{1}{t}\, dt = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k}}{z_0 + j} \binom{k}{j} \binom{z-z_0}{k+1} + \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z_0}\right). \tag{123}$$

The right-hand member of (123) is a finite sum when $z - z_0$ is an integer.

4.2. *The Stirling's Expansion with Remainder*

We examine the coefficients $c_{k,m}(n, \alpha)$ as $\alpha \to \infty$. From (16) and (81) we see that

$$\lim_{\alpha \to \infty} c_{k,m}(n, \alpha) = \frac{1}{k!\,(m-k)!}\, \Delta^n F_{k,m}\left(-\frac{n}{2}\right), \tag{124}$$

where

$$F_{k,m}(z) := \frac{z\Gamma\left(z + \frac{k}{2}\right)}{\Gamma\left(z + 1 - \frac{k}{2}\right)} \cdot \frac{z\Gamma\left(z + \frac{(m-k)}{2}\right)}{\Gamma\left(z + 1 - \frac{(m-k)}{2}\right)}.$$

Using the ∞-binomial formula (see (85)) where $a = b$ and a is replaced by ia, we obtain, with $w := \sin(\frac{a}{2})$,

$$(2i)^n \left(2w\sqrt{1-w^2}\right)^n = \sum_{m=0}^{\infty} \sum_{k=0}^{m} c_{k,m}(n, \infty)(2iw)^m, \tag{125}$$

whence

$$m!(2i)^m \sum_{k=0}^{m} c_{k,m}(n, \infty) = 2^{2n} i^n \frac{d^m}{dw^m} \left(w\sqrt{1-w^2}\right)^n \bigg|_{w=0}, \tag{126}$$

i.e.,

$$\sum_{k=0}^{m} c_{k,m}(n,\infty)=\begin{cases} 2^{2n-m}\begin{pmatrix} \frac{n}{2} \\ \frac{m-n}{2} \end{pmatrix} & \text{if } (m-n) \text{ is even} \\ 0 & \text{if } (m-n) \text{ is odd.} \end{cases} \tag{97}$$

We will be more precise for $n=1$, the case to be considered in the expansion (30).

LEMMA 4.1. *The coefficients* $c_{k,m}(1,\infty)$, $0 \leqslant k \leqslant m$, *are all equal to zero if* $k \neq 1$ *or* $(m-1)$. *Moreover,*

$$c_{1,m}(1,\infty)=c_{m-1,m}(1,\infty)=\begin{cases} \dfrac{1}{2^{m-1}}\begin{pmatrix} \frac{1}{2} \\ \frac{m-1}{2} \end{pmatrix} & \textit{if } m \textit{ is odd} \\ 0 & \textit{if } m \textit{ is even.} \end{cases} \tag{128}$$

Proof. The first part of the lemma is obtained by direct computation using (124); we have

$$\begin{aligned} k!(m-k)!\, c_{k,m}(1,\infty) &= \Delta F_{k,m}\left(-\frac{1}{2}\right) \\ &= F_{k,m}\left(\frac{1}{2}\right)-F_{k,m}\left(-\frac{1}{2}\right) \\ &= \left(\frac{1}{2}\right)^2 \frac{\Gamma\left(1+\frac{(k-1)}{2}\right)\Gamma\left(1+\frac{(m-k-1)}{2}\right)}{\Gamma\left(1-\frac{(k-1)}{2}\right)\Gamma\left(1-\frac{(m-k-1)}{2}\right)} \\ &\quad -\left(\frac{1}{2}\right)^2 \frac{\Gamma\left(\frac{(k-1)}{2}\right)\Gamma\left(\frac{(m-k-1)}{2}\right)}{\Gamma\left(-\frac{(k-1)}{2}\right)\Gamma\left(-\frac{(m-k-1)}{2}\right)} \\ &= 0 \qquad \text{if} \quad k \neq 1, (m-1) \quad \textit{since} \ \ \Gamma(z+1)=z\Gamma(z). \end{aligned}$$

Formula (128) follows at once from (127) since its left-hand member then reduced to $2c_{1,m}(1,\infty)$. ▮

Taking into account the above lemma, we deduce the following result.

COROLLARY 4.1. *The Stirling's expansion takes the form*

$$f(z)=\sum_{n=0}^{N}\Delta^{n}f\left(z_0-\frac{n}{2}\right)\frac{(z-z_0)\,\Gamma\left(z-z_0+\frac{n}{2}\right)}{n!\,\Gamma\left(z-z_0+1-\frac{n}{2}\right)}+R_{N,\infty}(z_0,z)$$

$$-\sum_{n=1}^{N}\sum_{k=1}^{\infty}\frac{c_{1,2k+1}(1,\infty)}{(n-1)!}\,{}_{\infty}\!\int_{z_0}^{z}\phi_{n-1,\infty}(z-t)\,\Delta^{n+2k}f\left(t-\frac{(n+2k)}{2}\right)dt$$

$$+\sum_{n=1}^{N}\sum_{k=1}^{[n/2]}\frac{c_{2k,2k+1}(1,\infty)}{(n-2k)!}$$

$$\times\,{}_{\infty}\!\int_{z_0}^{z}\phi_{n-2k,\infty}(z-t)\,\Delta^{n+1}f\left(t-\frac{(n+1)}{2}\right)dt, \tag{129}$$

where $c_{1,2k+1}(1,\infty)=c_{2k,2k+1}(1,\infty)=(-1)^{k-1}\binom{2k}{k}/2^{4k}(2k-1)$.

The particular case $N=1$ of (129) can be written as

$${}_{\infty}\!\int_{z_0}^{z}(z-t)\,\Delta^{2}f(t-1)\,dt=\sum_{k=0}^{\infty}\frac{(-1)^{k-1}\binom{2k}{k}}{2^{4k}\,(2k-1)}\left(\Delta^{2k}f(z-k)-\Delta^{2k}f(z_0-k)\right)$$

$$-\left(f\left(z_0+\frac{1}{2}\right)-f\left(z_0-\frac{1}{2}\right)\right)(z-z_0). \tag{130}$$

4.3. *The Fundamental Quantities* $d_{k,\alpha}^{(n)}$

We add a remark concerning the form of the coefficients $d_{k,\alpha}^{(n)}$ defined by (44). Empirical computations seem to indicate that these quantities can be written as

$$d_{n+2k,\alpha}^{(n)}=\frac{n(n+2k)!\,(2\alpha-1)\,P_{N(k)}(\alpha,n)}{d_k\prod_{j=1}^{k}(\alpha+j)^{[k/j]}} \tag{131}$$

for $k=1,2,3,\ldots$, where d_k is an integer depending only on k and $P_{N(k)}(\alpha, n)$ is a polynomial in α of degree $N(1):=0$ and $N(k):=\sum_{j=1}^{k-1}[k/j]$, $k=2, 3, \ldots$. For example, apart the values given in Section 2, we find that

$$d^{(n)}_{n+6,\alpha}=\frac{\begin{array}{c}n(n+6)!\,(2\alpha-1)((3012+1827n+210n^2)\\+(3080-1911n-665n^2)\,\alpha+(1808-2856n+175n^2)\,\alpha^2\\+(544-1176n+560n^2)\,\alpha^3+(64-168n+140n^2)\,\alpha^4)\end{array}}{23224320(\alpha+1)^3\,(\alpha+2)(\alpha+3)}.$$

The integers d_k grow very rapidly, as is seen with the calculations $d_1=48$, $d_2=23{,}040$, $d_3=23{,}224{,}320$, $d_4=22{,}295{,}347{,}200$, $d_5=11{,}771{,}943{,}321{,}600$, etc.

In this context, the recurrence relation (44) can be written as

$$d^{(n)}_{n+2k,\alpha}=-2\sum_{\ell=0}^{k}\binom{n+2k}{n-1+2\ell}B_{2k-2\ell+1,\alpha}\,d^{(n-1)}_{n-1+2\ell,\alpha}. \tag{132}$$

Using repeatedly this relation, we obtain

$$\frac{d^{(n)}_{n+2k,\alpha}}{(n+2k)!}=(-2)^n\sum_{\ell_1=0}^{\ell_0}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{n-1}=0}^{\ell_{n-2}}\prod_{j=1}^{n}\left(\frac{B_{2(\ell_{j-1}-\ell_j)+1,\alpha}}{(2(\ell_{j-1}-\ell_j)+1)!}\right), \tag{133}$$

where $\ell_0:=k$ and $\ell_n:=0$.

4.4. *Special Quantities*

The limiting case $\alpha\to\frac{1}{2}$ has a special character (we would also consider $\alpha\to-1,-2,\ldots$). Looking at a general pattern which occurs in our context, we may be interested to study quantities as $\lim_{\alpha\to 1/2}((B_{n,\alpha}(z)-B_n(z))/(\alpha-1/2))$, $\lim_{\alpha\to 1/2}((b_\alpha(f;z)-f'(z))/(\alpha-1/2))$, $\lim_{\alpha\to 1/2}(d^{(n)}_{n+2k,\alpha}/(\alpha-1/2))$, etc. (and similar limits as $\alpha\to-1,-2,\ldots$). More generally, we may want to consider such quantities as

$$\lim_{\alpha\to\beta}\frac{B_{n,\alpha}(z)-B_{n,\beta}(z)}{\alpha-\beta}=:B_{n,\beta,2}(z)$$

(the β-Bernoulli polynomials of the second kind),

$$\lim_{\alpha\to\beta}\frac{b^{(n)}_\alpha(f;z)-b^{(n)}_\beta(f;z)}{\alpha-\beta}=:b^{(n)}_{\beta,2}(f;z)$$

(the β-derivative of the second kind),

$$\lim_{\alpha\to\beta}\frac{b^{(n)}_{\alpha,2}(f;z)-b^{(n)}_{\beta,2}(f;z)}{\alpha-\beta}=:b^{(n)}_{\beta,3}(f;z)$$

(the β-derivative of the third kind), etc. For examples, we have $B_{2,\alpha,2}(z)=1/8(\alpha+1)^2$, $B_{3,\alpha,2}(z)=3(2z-1)/16(\alpha+1)^2$, $B_{4,1/2,3}(z)=(-49+1000z-1000z^2)/2250$, $b_{\alpha,2}(f;z)=(1/16(\alpha+1)^2)\,f'''(z)+((4\alpha^3+14\alpha^2-\alpha-23)/1536(\alpha+1)^3(\alpha+2)^2)\,f^{(\mathrm{v})}(z)+\cdots$, $b_{1/2,3}(f;z)=-(1/27)\,f'''(z)+(353/162000)\;f^{(\mathrm{v})}(z)-(18197/222264000)\,f^{(\mathrm{vii})}(z)+\cdots$, etc.

A situation where some kind of symmetry seems to happen is for the polynomials

$$\psi_n(z):=\lim_{\alpha\to 1/2}\frac{z^n-\phi_{n,\alpha}(z)}{\alpha-1/2}. \tag{134}$$

We have $\psi_0(z)=\psi_1(z)=\psi_2(z)=0$, $\psi_3(z)=z/6$, $\psi_4(z)=2z^2/3$, $\psi_5(z)=(z/180)(300z^2-13)$, $\psi_6(z)=(z^2/30)(100z^2-13)$, $\psi_7(z)=(z/1260)(7350z^4-1911z^2+94)$, $\psi_8(z)=(2z^2/315)(1470z^4-637z^2+94)$; these calculations and others indicate a possible relation of the form

$$\psi_{2m}(z)=c_{2m}z^2P_{m-2}(z^2),\qquad m=2,3,..., \tag{135}$$

$$\psi_{2m+1}(z)=d_{2m+1}zQ_{m-1}(z^2),\qquad m=1,2,..., \tag{136}$$

where $c_{2m}=2m\,d_{2m-1}$, $m=2,3,...$, and

$$\int_0^z Q_m(t^2)\,dt=zP_m(z^2), \tag{137}$$

for $m=0,1,2,...$.

Another situation is for the quantities

$$\gamma(k,m,1):=\lim_{\alpha\to 1/2}\frac{c_{k,m}(1,\alpha)}{\alpha-1/2}, \tag{138}$$

where $1\leqslant k\leqslant m$. We have $\gamma(k,m,1)=0$ for $m=2,4,6,...$. Letting $\gamma(m):=\gamma(1,m,1)$, we find that $\gamma(3)=\frac{1}{12}$, $\gamma(5)=-\frac{13}{4320}$ with $\gamma(2,5,1)=2\gamma(5)$, $\gamma(7)=\frac{47}{453600}$ with $\gamma(2,7,1)=3\gamma(7)$ and $\gamma(3,7,1)=5\gamma(7)$, $\gamma(9)=-\frac{1703}{508032000}$ with $\gamma(2,9,1)=4\gamma(9)$, $\gamma(3,9,1)=\frac{28}{3}\gamma(9)$ and $\gamma(4,9,1)=14\gamma(9)$, $\gamma(11)=\frac{7817}{75442752000}$ with $\gamma(2,11,1)=5\gamma(11)$, $\gamma(3,11,1)=15\gamma(11)$, $\gamma(4,11,1)=30\gamma(11)$ and $\gamma(5,11,1)=42\gamma(11)$.

4.5. *The Universal Character of the Exponential Function*

If $f(z)$ is an α-analytic function such that $b_\alpha(f; z) \equiv f(z)$ then we have $b_\alpha^{(n)}(f; z) \equiv f(z)$ for $n = 0, 1, 2, ...,$ whence

$$f(z) = \sum_{n=0}^{\infty} \frac{b_\alpha^{(n)}(f; z_0)}{n!} \phi_{n,\alpha}(z - z_0)$$

$$= f(z_0) \sum_{n=0}^{\infty} \frac{1}{n!} \phi_{n,\alpha}(z - z_0).$$

Let

$$e_\alpha(z) := \sum_{n=0}^{\infty} \frac{1}{n!} \phi_{n,\alpha}(z) \tag{139}$$

if the series converges. A simple expression is sometimes available for $e_\alpha(z)$. Consider the expansion (see (57))

$$e^{cz} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{e^{c/2} - e^{-c/2}}{g_\alpha\left(i \frac{c}{2}\right)} \right)^n \phi_{n,\alpha}(z), \tag{140}$$

which is valid for all z if $|c|$ is sufficiently small. Assume that the equation $e^{c/2} - e^{-c/2} = g_\alpha(i \frac{c}{2})$ has a solution $c = w_\alpha$ (numerical calculations indicate that a solution $0 < w_\alpha < 1$ seems to exist at least for positive values of α). If the expansion (140) is valid for the value $c = w_\alpha$ then we see that

$$e_\alpha(z) = e^{w_\alpha z}. \tag{141}$$

For instances, we have $e_{1/2}(z) = e^z$, $e_\infty(z) = ((\sqrt{5} + 1)/2)^{2z}$, and $e_{-1/2}(z) = 3^z$.

4.5.1. *A Representation for Bessel Functions.* We take $z_0 = -\frac{1}{2}$ and $z = \frac{1}{2}$ in (106). The resulting formula holds for small $|c|$. Replacing c by the new variable $2iz$ gives the following result.

PROPOSITION 4.1. *Let z be a complex number such that $|z|$ is sufficiently small. We have the representation*

$$\frac{J_\alpha(z)}{z^\alpha} = \frac{1}{2^\alpha \Gamma(\alpha + 1)} \, {}_\alpha\!\int_{-1/2}^{1/2} e^{2izt} \, dt. \tag{142}$$

For $\alpha = \frac{1}{2}$ a cancellation occurs during the proof of (142) and the formula is of course valid for all complex numbers z:

$$\frac{\sin(z)}{z} = \int_{-1/2}^{1/2} e^{2izt}\, dt. \tag{143}$$

The case $\alpha = -\frac{1}{2}$, namely

$$\cos(z) = {}_{-1/2}\int_{-1/2}^{1/2} e^{2izt}\, dt, \tag{144}$$

is an immediate consequence of (28). The limiting case $\alpha \to \infty$ gives a relation which is readily seen to be equivalent to a binomial formula:

$$\sqrt{1-u} = 1 - \sum_{k=1}^{\infty} \frac{\binom{2k}{k} u^k}{2^{2k}(2k-1)}, \qquad 0 \leqslant u \leqslant 1. \tag{145}$$

4.5.2. *A Generalized Orthogonality Property.* The orthogonality of the set of functions $\{e^{2n\pi it} : n \in \mathbb{Z}\}$, on the interval (0, 1), is equivalent to the relation

$$\int_0^1 e^{2k\pi it}\, dt = \begin{cases} 0, & k \in \mathbb{Z}, \quad k \neq 0 \\ 1, & k = 0. \end{cases} \tag{146}$$

Let $c = (2k\pi i/(z - z_0)$ in (106), $k \neq 0$ an integer, where $z - z_0 \neq \frac{k}{\ell}$ for an integer ℓ. Then the numerator in the right-hand member of (106) vanishes and the denominator does not. We mention the

PROPOSITION 4.2. *Let z_0, z be complex numbers and k be an integer such that $0 < |k| \leqslant c(\alpha)\, |z - z_0|$ for some sufficiently small positive constant $c(\alpha)$. Then we have*

$$_{\alpha}\int_{z_0}^{z} e^{2k\pi it/z - z_0}\, dt = 0. \tag{147}$$

5. FINAL REMARKS

A basic idea in order to obtain concrete applications of the α-calculus could be to write first the classical formulas in the α-extended form. Then a typical situation is that we can isolate a factor $(\alpha - \frac{1}{2})$ (or $(\alpha - \beta)$) and after suitable passages to the limits as $\alpha \to \frac{1}{2}$ (or $\alpha \to \beta$) we get new formulas which are hoped to be useful. We illustrate the kind of procedure that may be followed with an example. For $\tau < 2\min(|j_1(\alpha)|, |j_1(\beta)|)$, formula (2) implies that

$$\sum_{k=0}^{\infty} \frac{(B_{k,\beta} - B_{k,\alpha})}{k!} (f^{(k)}(1) - (-1)^k f^{(k)}(0)) = 0. \tag{148}$$

Dividing both sides of (148) by $(\beta - \alpha)$ and letting $\beta \to \alpha$, we obtain

$$\sum_{k=0}^{\infty} \frac{B_{k,\alpha,2}}{k!} (f^{(k)}(1) - (-1)^k f^{(k)}(0) = 0. \tag{149}$$

The relation (149) is also valid for any polynomial. For $f(z) = z^j$, $j = 2\ell + 1$, we obtain the recurrence relation

$$B_{2\ell+1,\alpha,2} = -\frac{1}{2} \sum_{k=0}^{2\ell} \binom{2\ell+1}{k} B_{k,\alpha,2} \tag{150}$$

for $\ell = 0, 1, 2, \ldots$.

We wish also to consider the apparent possibility to define a more general operator than $b_\alpha(\,; z)$. Let

$$b_\alpha(x; f; z) := \sum_{k=0}^{\infty} \frac{(B_{k,\alpha}(x+1) - B_{k,\alpha}(x))}{k!} f^{(k)}(z), \tag{151}$$

so that $b_\alpha(f; z) = b_\alpha(0; f; z)$. Denote the right-hand member of (151) by $F(x)$. We have $F^{(\ell)}(0) = b_\alpha(f^{(\ell)}; z)$ and thus

$$\begin{aligned} F(x) &= \sum_{\ell=0}^{\infty} \frac{F^{(\ell)}(0)}{\ell!} x^\ell \\ &= \sum_{\ell=0}^{\infty} \frac{b_\alpha(f^{(\ell)}; z)}{\ell!} x^\ell \\ &= b_\alpha\left(\sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(z)}{\ell!} x^\ell ; z \right) \\ &= b_\alpha(f(x+z); z), \end{aligned}$$

i.e.,

$$b_\alpha(x; f; z) = b_\alpha(f(\zeta); \zeta = x + z). \tag{152}$$

Other elements of interest of the α-calculus are first the possibility to define consistently the double α-integral. An elementary geometric construction permits also to define correctly the curvilinear α-integral. A generalized version of Green–Riemann Theorem is then available:

$$ {}_{\alpha}\oint_{C} F(u,v)\,du + G(u,v)\,dv $$

$$ = {}_{\alpha}\int {}_{\alpha}\int_{D} (b_{\alpha}(G(u,v);u) - b_{\alpha}(F(u,v);v))\,du\,dv. \tag{153} $$

The complex α-calculus appears in a natural way. An α-extension of Cauchy Theorem,

$$ \oint_{C} f(z)\,dz = 0, \tag{154} $$

can be proved in the appropriate context. These points will be the main purpose of a subsequent paper.

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